Brownian motion and Stochastic Calculus
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Assignment 10—solutions

Exercise 1

We fix a standard one-dimensional $(\mathbb{F},\mathbb{P})$–Brownian motion.

1) Show that for any $C^{1,2}$ function $f:[0,+\infty)\times\mathbb{R}\longrightarrow\mathbb{R}$, such that there exists some continuous function $C:[0,+\infty)\longrightarrow[0,+\infty)$ with
$$|\partial_x f(t,x)|\leq C(t)\mathrm{e}^{C(t)|x|},\ (t,x)\in[0,+\infty)\times\mathbb{R}, \tag{0.1}$$
the process $(f(t,B_t))_{t\geq 0}$ will be an $(\mathbb{F},\mathbb{P})$-martingale if and only if
$$\partial_t f(t,x)+\frac{1}{2}\partial^2_{xx}f(t,x)=0,\ (t,x)\in[0,+\infty)\times\mathbb{R}. \tag{0.2}$$

2) in this question, we are looking for functions f of the form
$$f(t,x)=\sum_{i=0}^n\sum_{j=0}^n a_{i,j}t^ix^j,\ (t,x)\in[0,+\infty)\times\mathbb{R},$$
for some integer n and real numbers $(a_{i,j})_{(i,j)\in\{0,\dots,n\}^2}$. Show that the process $f(t,B_t)$ is an $(\mathbb{F},\mathbb{P})$-martingale if and only if the $(a_{0,j})_{j\in\{0,\dots,n\}}$ are arbitrarily fixed and
$$\begin{cases} a_{i,j}=(-1)^i\dfrac{(j+2i)!}{2i!j!}a_{0,j+2i},\ j+2i\leq n,\\ a_{i,j}=0,\ j+2i>n, \end{cases}$$

1) **Indeed, by Itô's formula the Itô process $(f(t,B_t))_{t\geq 0}$ is an $(\mathbb{F}^{B,\mathbb{P}},\mathbb{P})$–local martingale if and only if its drift is equal to 0 with $\mathbb{P}$-probability 1, that is**
$$\partial_t f(t,B_t)+\frac{1}{2}\partial^2_{xx}f(t,B_t)=0,\ t\geq 0,\ \mathbb{P}\text{–a.s.}$$
Since the support of the $\mathbb{P}$-distribution of B_t is $\mathbb{R}$, we deduce the desired condition. Next, since inequality (0.1) holds, the volatility of $(f(t,B_t))_{t\geq 0}$ is automatically in $\mathbb{H}^2(\mathbb{R},\mathbb{F}^{B,\mathbb{P}},\mathbb{P})$, which shows the martingale property.

2) **In this case, inequality (0.1) is obviously satisfied, and direct computations prove that (0.2) holds if and only if $a_{1,0}=a_{1,1}=0$ when $n=1$ (the case $n=0$ is trivial), and when $n\geq 2$**
$$\begin{cases} a_{i+1,j}=-\dfrac{(j+2)(j+1)}{2(i+1)}a_{i,j+2},\ i\in\{0,\dots,n-1\},\ j\in\{0,\dots,n-2\},\\ a_{n,j+2}=0,\ j\in\{0,\dots,n-2\},\\ a_{i+1,n-1}=0,\ i\in\{0,\dots,n-1\},\\ a_{i+1,n}=0,\ i\in\{0,\dots,n-1\}. \end{cases}$$
It can then easily be checked that this equivalent to having the $(a_{0,j})_{j\in\{0,\dots,n\}}$ arbitrarily fixed and
$$\begin{cases} a_{i,j}=(-1)^i\dfrac{(j+2i)!}{2i!j!}a_{0,j+2i},\ j+2i\leq n,\\ a_{i,j}=0,\ j+2i>n, \end{cases}$$

Exercise 2

Consider, for any $x \in \mathbb{R}^d$, the SDE

$$\mathrm{d}X_t^x = a(X_t^x)\mathrm{d}t + b(X_t^x)\mathrm{d}W_t,\ X_0^x = x,$$

where W is a $\mathbb{R}^m$-valued Brownian motion, $a : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $b : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times m}$ are measurable and locally bounded. We fix a non-empty, bounded open subset U of $\mathbb{R}^d$ and assume that for any $x \in U$, we have with $T_U^x := \inf\{s \geq 0 : X_s^x \notin U\}$, that T_U^x is $\mathbb{P}$-integrable.

Moreover, consider the boundary problem

$$Lu(x) + c(x)u(x) = -f(x),\ \text{for } x \in U,\ u(x) = g(x),\ \text{for } x \in \partial U,$$

where $f \in C_b(U)$, $g \in C_b(\partial U)$, $c \leq 0$ is a uniformly bounded function on $\mathbb{R}^d$, and L is defined by

$$Lf(x) := \sum_{i=1}^d a^i(x)\frac{\partial f}{\partial x^i}(x) + \frac{1}{2}\sum_{(i,j)\in\{1,\dots,d\}^2}\left(bb^\top\right)^{ij}(x)\,\frac{\partial^2 f}{\partial x^i\,\partial x^j}(x).$$

Show that if $u \in C^2(U) \cap C(\bar{U})$ is a solution of the above boundary problem and $(X_t^x)_{t\geq 0}$ is a solution of the SDE for some $x \in U$, then

$$u(x) = \mathbb{E}^\mathbb{P}\left[g(X_{T_U^x}^x)\exp\left(\int_0^{T_U^x} c(X_s^x)\mathrm{d}s\right)\right] + \mathbb{E}^\mathbb{P}\left[\int_0^{T_U^x} f(X_s^x)\exp\left(\int_0^s c(X_r^x)\mathrm{d}r\right)\mathrm{d}s\right].$$

For $m \in \mathbb{N}^\star$ large enough so that $\frac{1}{m} < d(x, U^c)$, we define

$$T_m := \inf\left\{s \geq 0 : d(X_s^x, U^c) \leq 1/m\right\},$$

and construct $u_m \in C_c^2(\mathbb{R}^d, \mathbb{R})$ such that $u = u_m$ on $\left\{z \in U : d(z, U^c) \geq 1/m\right\}$. We apply Itô's formula to $u_m(X_t^x)\exp\left(\int_0^t c(X_s^x)\mathrm{d}s\right)$, take then the expectation and use that the local martingale is a true martingale as b is locally bounded and $u \in C_c^2$ to obtain that (for more details, see the end of this document)

$$\mathbb{E}^\mathbb{P}\left[u_m(X_{t\wedge T_m^x}^x)\exp\left(\int_0^{t\wedge T_m^x} c(X_s^x)\mathrm{d}s\right)\right] - u_m(x) = \mathbb{E}^\mathbb{P}\left[\int_0^{t\wedge T_m^x}\left(Lu_m(X_s^x) + c(X_s^x)u_m(X_s^x)\right)\exp\left(\int_0^s c(X_r^x)\mathrm{d}r\right)\mathrm{d}s\right].$$

Now, as $u_m = u$ on $\left\{z \in U : d(z, U^c) \geq \frac{1}{m}\right\}$, by definition of T_m^x and as u is the solution of the boundary problem, we obtain that

$$u(x) = \mathbb{E}\mathbb{P}\left[u(X_{t\wedge T_m^x}^x)\exp\left(\int_0^{t\wedge T_m^x} c(X_s^x)\mathrm{d}s\right)\right] + \mathbb{E}^\mathbb{P}\left[\int_0^{t\wedge T_m^x} f(X_s^x)\exp\left(\int_0^s c(X_r^x)\mathrm{d}r\right)\mathrm{d}s\right].$$

Since $T_m^x \uparrow T_U^x < \infty$, we can let $t \to \infty$ and then $m \to \infty$ to conclude, by the dominated convergence theorem, that

$$u(x) = \mathbb{E}^\mathbb{P}\left[g(X_{T_U^x}^x)\exp\left(\int_0^{T_U^x} c(X_s^x)\mathrm{d}s\right)\right] + \mathbb{E}^\mathbb{P}\left[\int_0^{T_U^x} f(X_s^x)\exp\left(\int_0^s c(X_r^x)\mathrm{d}r\right)\mathrm{d}s\right].$$

Exercise 3

Let $(B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion.

1) Show that the SDE

$$X_t = x + \int_0^t \sqrt{1 + X_s^2}\mathrm{d}B_s + \frac{1}{2}\int_0^t X_s\mathrm{d}s, \tag{0.3}$$

admits a unique strong solution for all $x \in \mathbb{R}$.

2) Fix $x \in \mathbb{R}$ and $(\beta_t, \gamma_t)_{t\geq 0}$ two independent one-dimensional Brownian motions. Show that

$$Y_t := \exp(\beta_t)\bigg(x + \int_0^t \exp(-\beta_s)\mathrm{d}\gamma_s\bigg),\ t \geq 0,$$

is well-defined and solves (0.3) for some well-chosen Brownian motion B. Deduce that for $a := \operatorname{argsinh}(x)$,

$$(Y_t,\ t \geq 0) \overset{\text{(law)}}{=} (\sinh(a + B_t),\ t \geq 0).$$

3) We now go to a slightly more general setting.

a) Show that if the map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^2 diffeomorphism from $\mathbb{R}$, then $\Phi_t := \varphi(B_t)$ satisfies

$$\Phi_t = \varphi(0) + \int_0^t \sigma(\Phi_s)\mathrm{d}B_s + \int_0^t b(\Phi_s)\mathrm{d}s, \tag{0.4}$$

where

$$\sigma(x) := (\varphi' \circ \varphi^{(-1)})(x),\ b(x) := \frac{1}{2}(\varphi'' \circ \varphi^{(-1)})(x).$$

b) Conversely, if $\sigma, b : \mathbb{R} \longrightarrow \mathbb{R}$ are Lipschitz functions with appropriate growth, we know that the SDE (0.4) admits a unique strong solution. Under which conditions on (σ, b) can we solve the system

$$\varphi'(y) = \sigma(\varphi(y)),\ \varphi''(y) = 2b(\varphi(y)),$$

so that the solution of (0.4) is $\Phi_t = \varphi(B_t)$?

1) **The drift and volatility are clearly Lipschitz-continuous with linear growth, hence the standard Cauchy–Lipschitz theorem applies.**

2) **We have for any** $t \geq 0$

$$\mathbb{E}^{\mathbb{P}}\bigg[\int_0^t \mathrm{e}^{-2\beta_s}\mathrm{d}s\bigg] = \int_0^t \mathrm{e}^{2s}\mathrm{d}s < +\infty,$$

ensuring that Y **is well-defined. Next, applying Itô's formula to** Y**, we get**

$$\mathrm{d}Y_t = \bigg(x + \int_0^t \exp(-\beta_s)\mathrm{d}\gamma_s\bigg)\mathrm{e}^{\beta_t}\bigg(\mathrm{d}\beta_t + \frac{1}{2}\mathrm{d}t\bigg) + \mathrm{d}\gamma_t = \frac{1}{2}Y_t\mathrm{d}t + \sqrt{1+Y_t^2}\bigg(\frac{Y_t}{\sqrt{1+Y_t^2}}\mathrm{d}\beta_t + \frac{1}{\sqrt{1+Y_t^2}}\mathrm{d}\gamma_t\bigg).$$

Now, let $B := \int_0^\cdot \big(\frac{Y_t}{\sqrt{1+Y_t^2}}\mathrm{d}\beta_t + \frac{1}{\sqrt{1+Y_t^2}}\mathrm{d}\gamma_t\big)$**. We have**

$$[B]_t = t,$$

ensuring by Lévy's characterisation that B **is a Brownian motion.**

Next, it is direct to check that $X_t := (\sinh(a + B_t)$ **is the strong solution to the SDE, which gives the desired result by uniqueness in law.**

3)*a)* **It is a simple application of Itô's formula. Indeed, we have**

$$\Phi_t = \Phi_0 + \int_0^t \varphi'(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t \varphi''(B_s)\mathrm{d}s,$$

and it suffices to notice that $B_t = \varphi^{(-1)}(\Phi_t)$**,** $t \geq 0$**.**

b) **If we can find** φ **as a** C^2 **diffeomorphism satisfying the two ODEs, then clearly** $\Phi = \varphi(B)$**. Now, it is necessary for this that**

$$\varphi'(y)\sigma'(\varphi(y)) = \sigma(\varphi(y))\sigma'(\varphi(y)) = 2b(\varphi(y)).$$

Hence, φ being a diffeomorphism on $\mathbb{R}$, this means that we must have

$$\sigma\sigma' = 2b.$$

Under this assumption, and if for instance $\sigma\sigma'$ is Lipschitz-continuous, and σ has a fixed sign, the result will hold.

How to apply Itô's formula in Exercise 2?

We want to apply Itô's formula to

$$u_m(X_t^x)\exp\left(\int_0^t c(X_s^x)ds\right) = f(Y_t, Z_t),$$

where $f : \mathbb{R}^{d+1} \to \mathbb{R}$, $f(y,z) = u_m(y)\exp(z)$, $Y_t = X_t^x$ and $Z_t = \int_0^t c(X_s^x)ds$. In particular, note that $Y_t = (Y_t^1, \dots, Y_t^d)$ is an $\mathbb{R}^d$-valued process, while Z_t is an $\mathbb{R}$-valued process. Hence, we have

$$\nabla f(y,z) = \begin{pmatrix} \nabla u_m(y)\exp(z) \\ u_m(y)\exp(z) \end{pmatrix} = \exp(z)\begin{pmatrix} \nabla u_m(y) \\ u_m(y) \end{pmatrix} \in \mathbb{R}^{d+1},$$

$$D^2 f(y,z) = \begin{pmatrix} D^2 u_m(y)\exp(z) & \nabla u_m(y)\exp(z) \\ (\nabla u_m(y))^T\exp(z) & u_m(y)\exp(z) \end{pmatrix} = \exp(z)\begin{pmatrix} D^2 u_m(y) & \nabla u_m(y) \\ (\nabla u_m(y))^T & u_m(y) \end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)},$$

and

$$\begin{pmatrix} dY_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} a(Y_t) \\ c(Y_t) \end{pmatrix} dt + \begin{pmatrix} b(Y_t) \\ 0 \end{pmatrix} dW_t.$$

Therefore, the drift vector of (Y_t, Z_t) is $(a(Y_t), c(Y_t))$ and the diffusion matrix is $(b(Y_t), 0)$. In particular, the quadratic covariation matrix is given by

$$\begin{pmatrix} b(Y_t) \\ 0 \end{pmatrix}\begin{pmatrix} b(Y_t)^T & 0 \end{pmatrix} = \begin{pmatrix} b(Y_t)b(Y_t)^T & 0 \\ 0 & 0 \end{pmatrix},$$

i.e., $\mathrm{d}[Y^i, Y^j]_t = (b(Y_t)b(Y_t)^T)^{ij}$ and $\mathrm{d}[Z,Z]_t = \mathrm{d}[Y^i, Z]_t = 0$, for any $i,j = 1,\dots,d$.

Now, recall Itô's formula (see Theorem 3.4.1, lecture notes) and note that if we have a function $g : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ (C^1 in time and C^2 in space) and an $\mathbb{R}^n$-valued Itô process $dX_t = \mu_t dt + \sigma_t dW_t$, then Itô's formula can be written as follows

$$g(t,X_t) = g(0,X_0) + \int_0^t \left[\partial_t g(s,X_s) + (\nabla_x g(s,X_s))^T\mu_t + \frac{1}{2}\sum_{i,j=1}^n (\sigma_t\sigma_t^T)^{ij}(D_x^2 g(s,X_s))^{ij}\right] dt + \int_0^t (\nabla_x g(s,X_s))^T\sigma_t dW_t,$$

where $\nabla_x g(s,X_s)$ is the vector of the spacial first derivatives and $D_x^2 g(s,X_s)$ the matrix of the spacial second derivatives. Then, let us apply Itô's formula to the function f and the $\mathbb{R}^{d+1}$-valued process (Y_t, Z_t). As the function f does not depend on time, we get

$$\begin{aligned}
f(Y_t,Z_t) &= f(Y_0,Z_0) + \int_0^t \left[(\nabla f(Y_s,Z_s))^T\begin{pmatrix} a(Y_t) \\ c(Y_t) \end{pmatrix} + \frac{1}{2}\sum_{i,j=1}^{d+1}\left(\begin{pmatrix} b(Y_t)b(Y_t)^T & 0 \\ 0 & 0 \end{pmatrix}\right)^{ij}(D^2 f(Y_s,Z_s))^{ij}\right] dt \\
&\quad + \int_0^t (\nabla f(Y_s,Z_s))^T\begin{pmatrix} b(Y_t) \\ 0 \end{pmatrix} dW_t \\
&= f(Y_0,Z_0) + \int_0^t \exp(Z_s)\left[(\nabla u_m(Y_s))^T a(Y_s) + u_m(Y_s)c(Y_s) + \frac{1}{2}\sum_{i,j=1}^d (b(Y_t)b(Y_t)^T)^{ij}(D^2 u_m(Y_s))^{ij}\right] ds \\
&\quad + \int_0^t \exp(Z_s)(\nabla u_m(Y_s))^T b(Y_t)dW_t \\
&= u_m(Y_0) + \int_0^t \exp(Z_s)\left[Lu_m(Y_s) + u_m(Y_s)c(Y_s)\right] ds + \int_0^t \exp(Z_s)(\nabla u_m(Y_s))^T b(Y_t)dW_t,
\end{aligned}$$

where L is the Dynkin operator defined in Ex. 2.